# Beam Elements
## Are they precise ?

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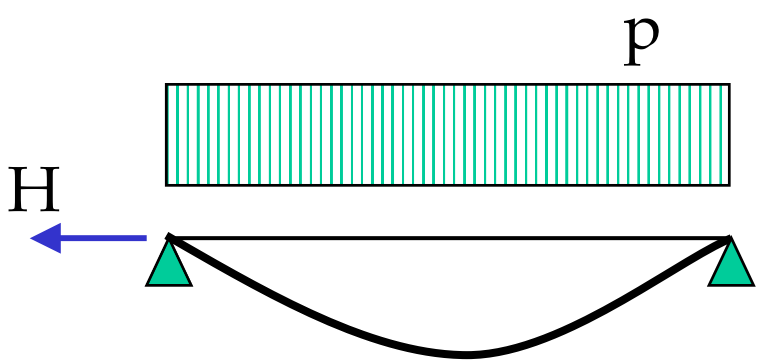
# Cable Analysis

**Cable**

$$-H\, w''(x) = p(x)$$

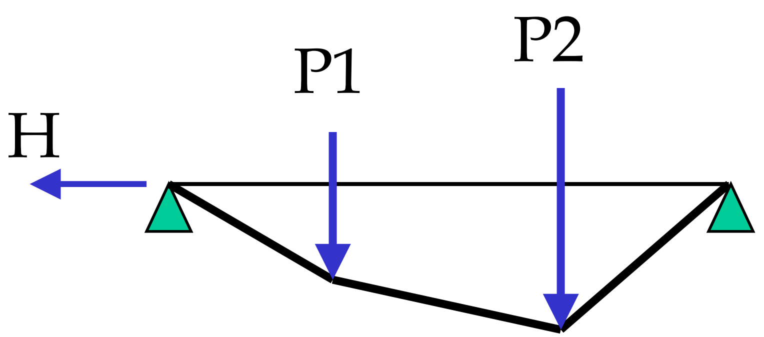



Solution for uniform load is a quadratic parabola or a sinh

**Cable element**

Linear between the nodes

Solution space: polygonal



# Cable Analysis

- Solution Space

  $w_h(x) = w_1 j_1(x) + w_2 j_2(x) + \ldots$
- Solution is exact for:

  Single forces
- Other loadings:

  Minimum of energy => equilibrium
- Exact solution is obtained in the nodes!

  Why ?

# Green's Functions

- The influence function (Green's function) for the deformation of the cable at a point is the cable polygon created by a point load P=1 at the point of interest.
- If the FE-System is able to model this solution exactly, the solution will be exact for this value.
- In all other cases, i.e. if the Green's function is not within the possible solution space, one has an approximate solution only.

# What are Beam Elements ?

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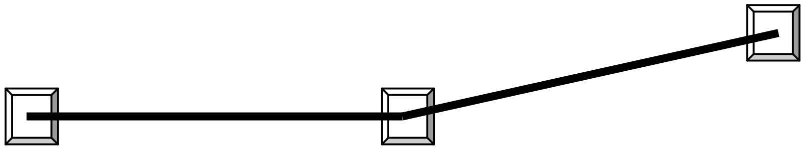
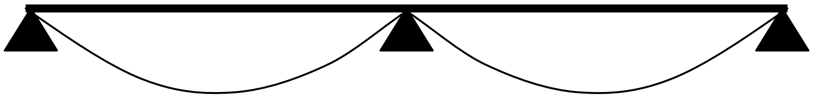
- A 3D Continua with a length >> width / height
- Simplification of the solution space (Bernoulli-Hypothesis, persistence of shape)
- Simplification for manual analysis (gravity centre, principal axis, shear etc.)
- Superstition:
  - Beam elements are simple
  - Beam elements are exact

# Problematic continuous beam

- Mechanic is not consistent



Plan view



Moments

# Sections

- **Normal stress**

$$u = u_0 + \varphi_y z - \varphi_z y$$

$$\sigma_x = E \varepsilon_x = E \frac{\partial u}{\partial x} = E \left[ \frac{\partial u}{\partial x} + \frac{\partial \varphi_y}{\partial x} z - \frac{\partial \varphi_z}{\partial x} y \right]$$

# Sections – Shear Stress

- V is only valid if the normal force and the section are constant along the axis
- Z is only valid if the section is not multiple connected
- The shear stress is not necessarily constant along the width b

$$\tau = \frac{V}{I}\frac{Z}{b}$$

- For biaxial bending or non effective parts of the section, the equation becomes more complex (Swain‘s formula)

# An other strategy: a deformation method

$$\tau_{xy} = G\left(\frac{\partial w}{\partial y} - z\frac{\partial \Theta_x}{\partial x}\right)$$

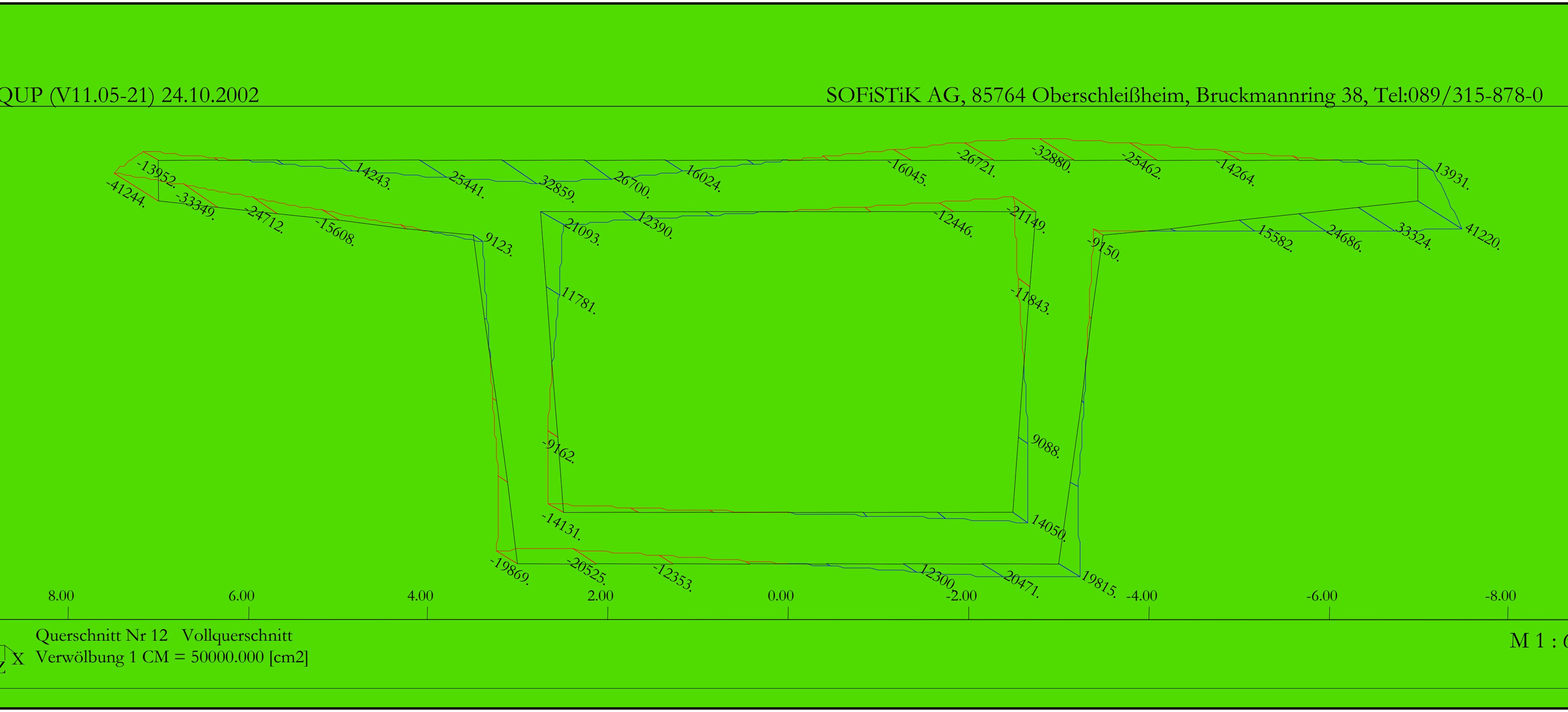
$$\tau_{xz} = G\left(\frac{\partial w}{\partial z} + y\frac{\partial \Theta_x}{\partial x}\right)$$

$$G\Delta w = G\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = -\frac{\partial \sigma_x}{\partial x}$$

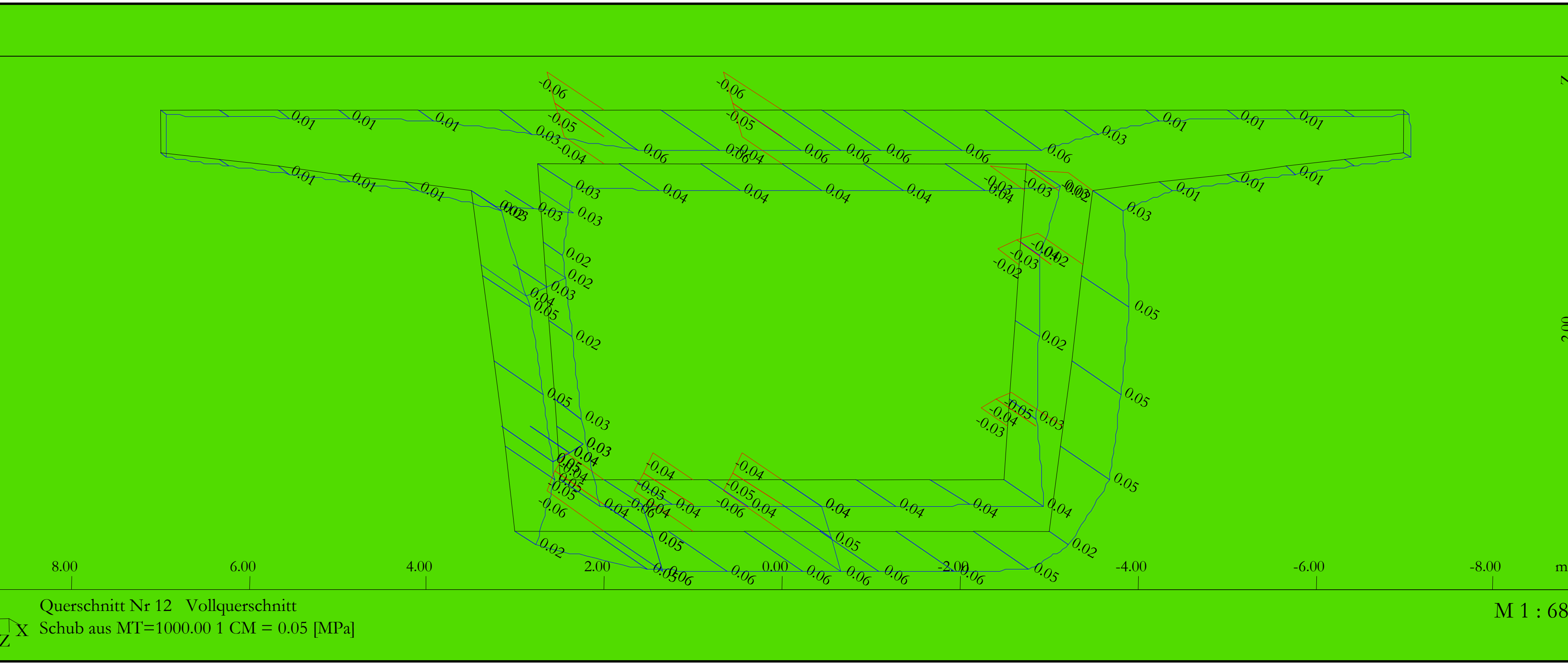
*Boundary Conditions:*

$$\tau_{xy} n_y + \tau_{xz} n_z = 0$$

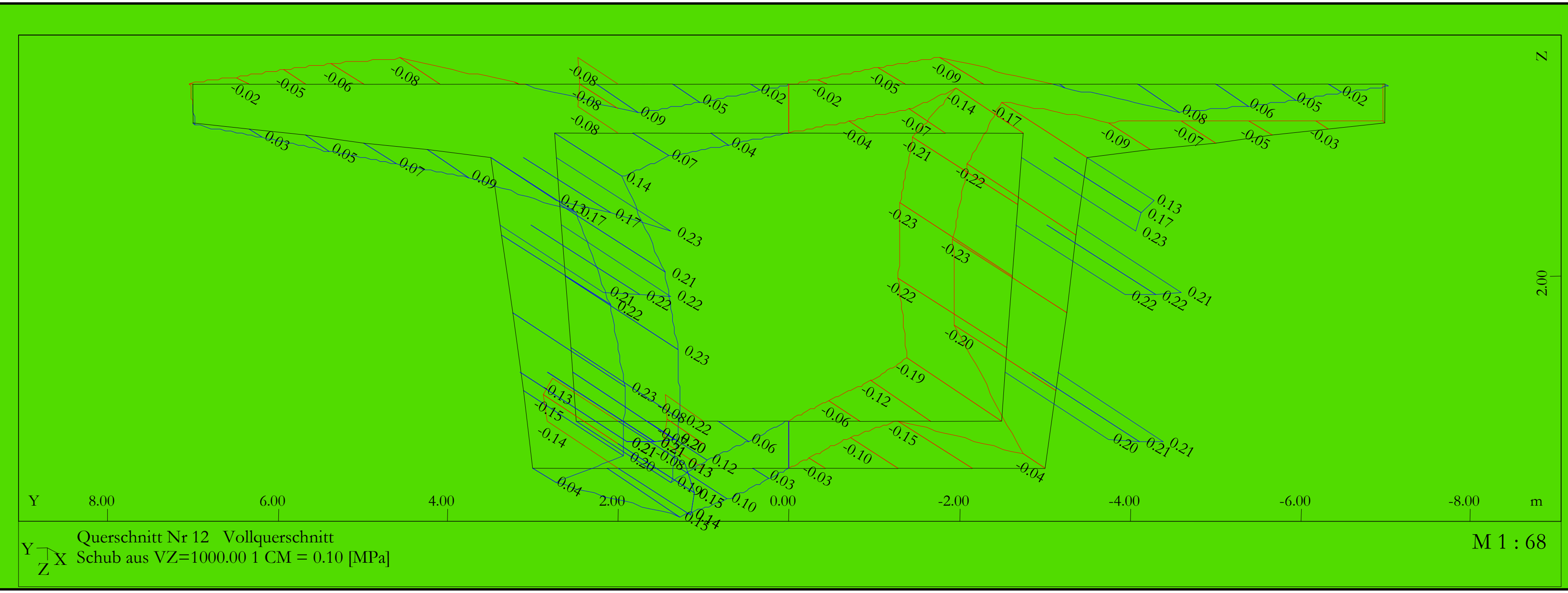
# Unit Warping



# Shear stress from Mt



# Shear stress from Vz



# Shear deformation

- Theory of Timoshenko/Marguerre (not exakt!)

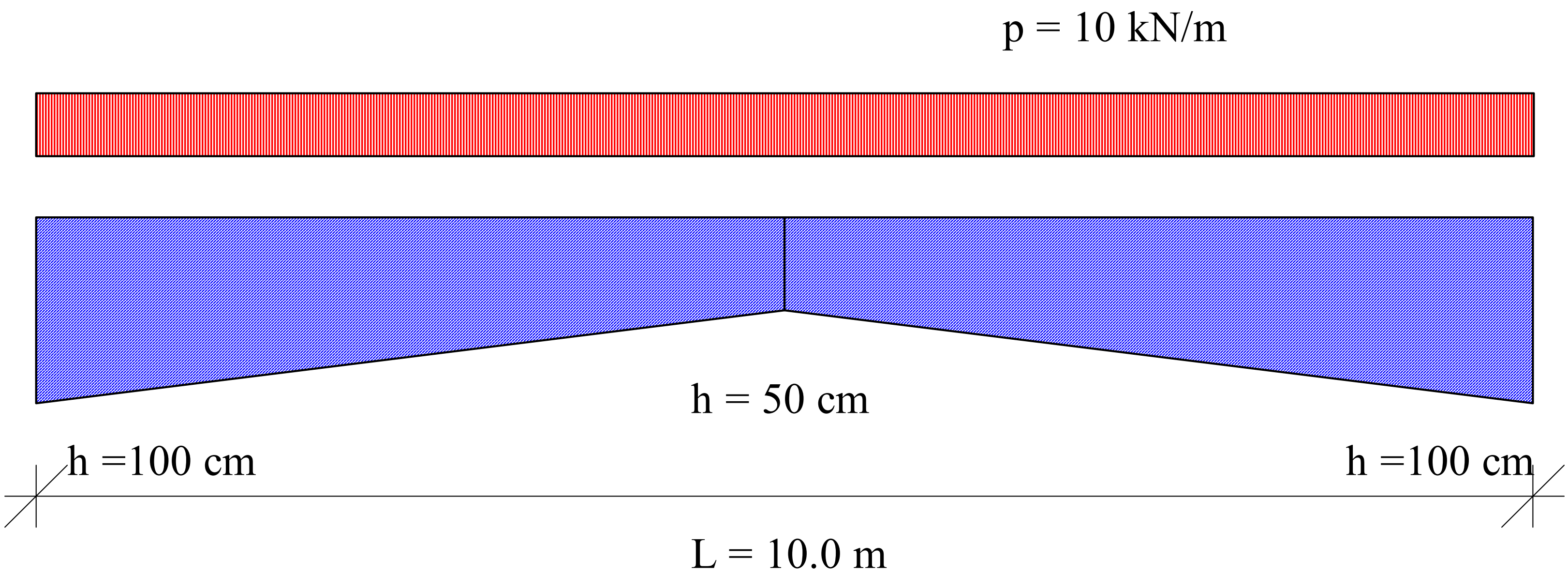
$$\Theta_y = \frac{V_z}{G\,A_z}$$

$$\varphi_y = \frac{\partial w}{\partial x} + \Theta_y$$

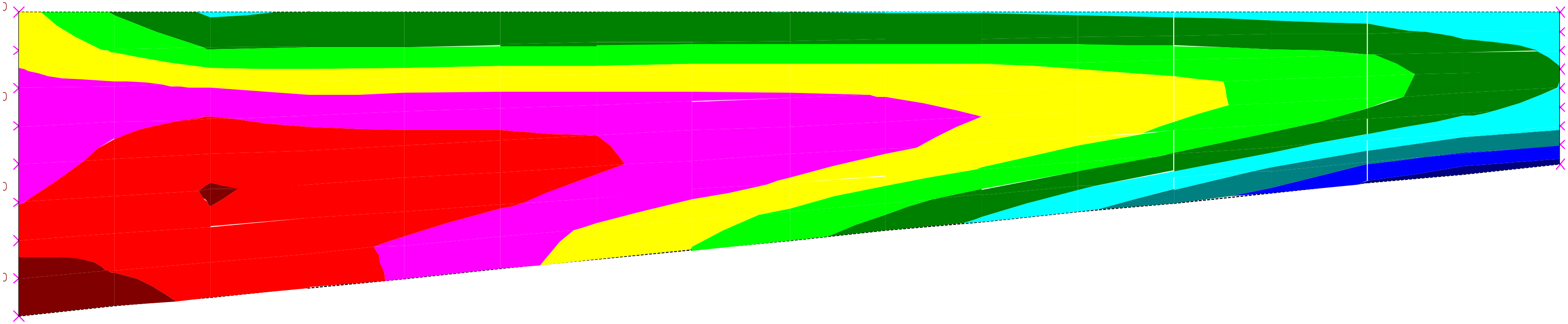
- Principal Axis of shear deformation

$$\begin{bmatrix} \Theta_y \\ \Theta_z \end{bmatrix} = \begin{bmatrix} \frac{1}{GA_y} & \frac{1}{GA_{yz}} \\ \frac{1}{GA_{yz}} & \frac{1}{GA_z} \end{bmatrix} * \begin{bmatrix} V_y \\ V_z \end{bmatrix}$$

# Shear in haunched beams

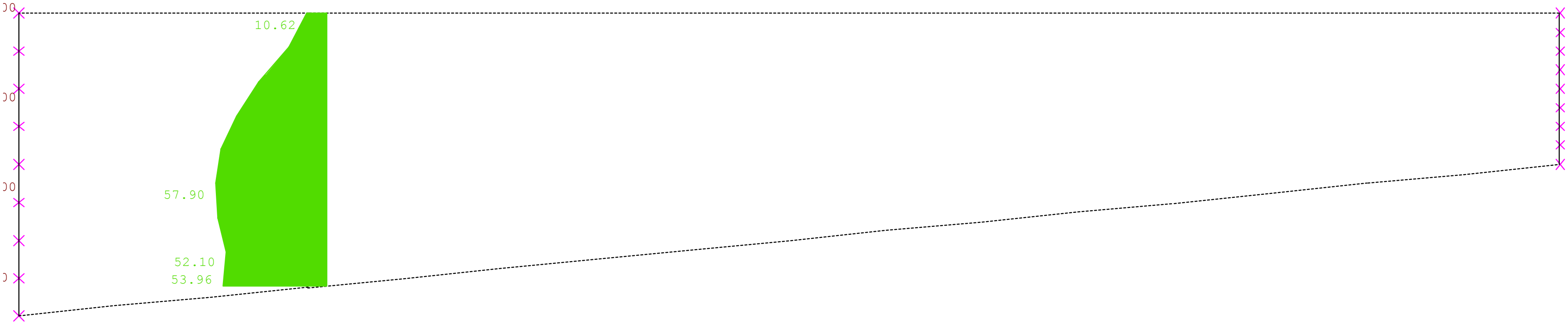


# FE-Analysis



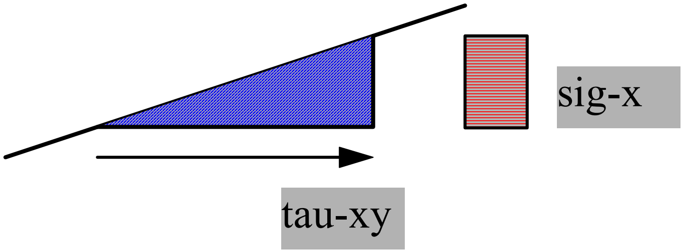
tau-xy

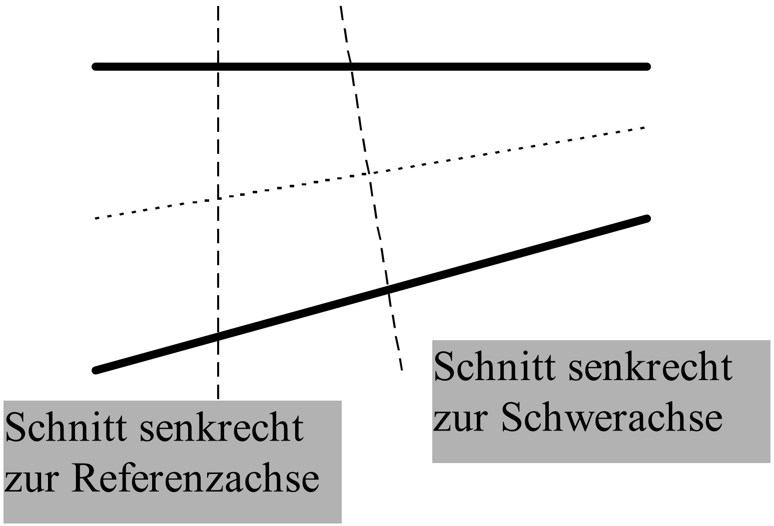
# FE-Analysis



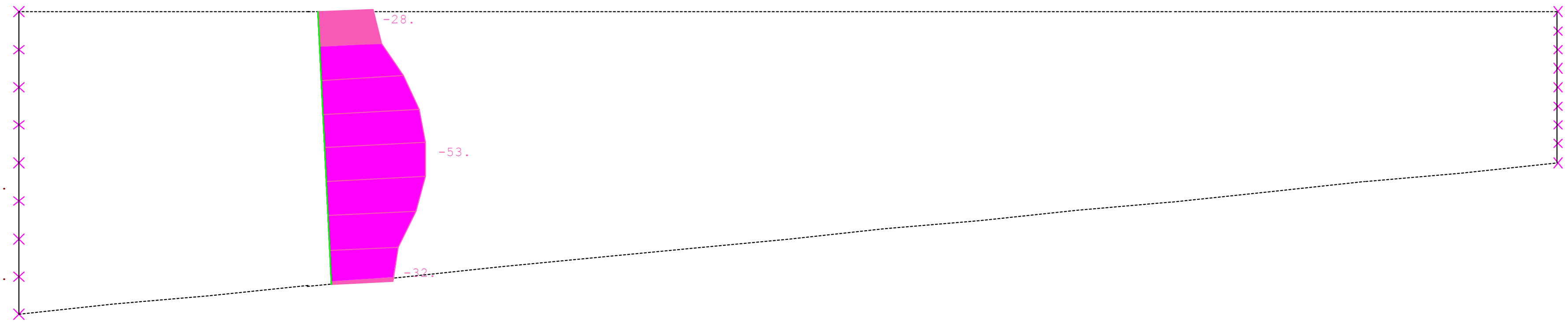

tau-xy in vertical section

# Shear in haunches



# FE-Analysis

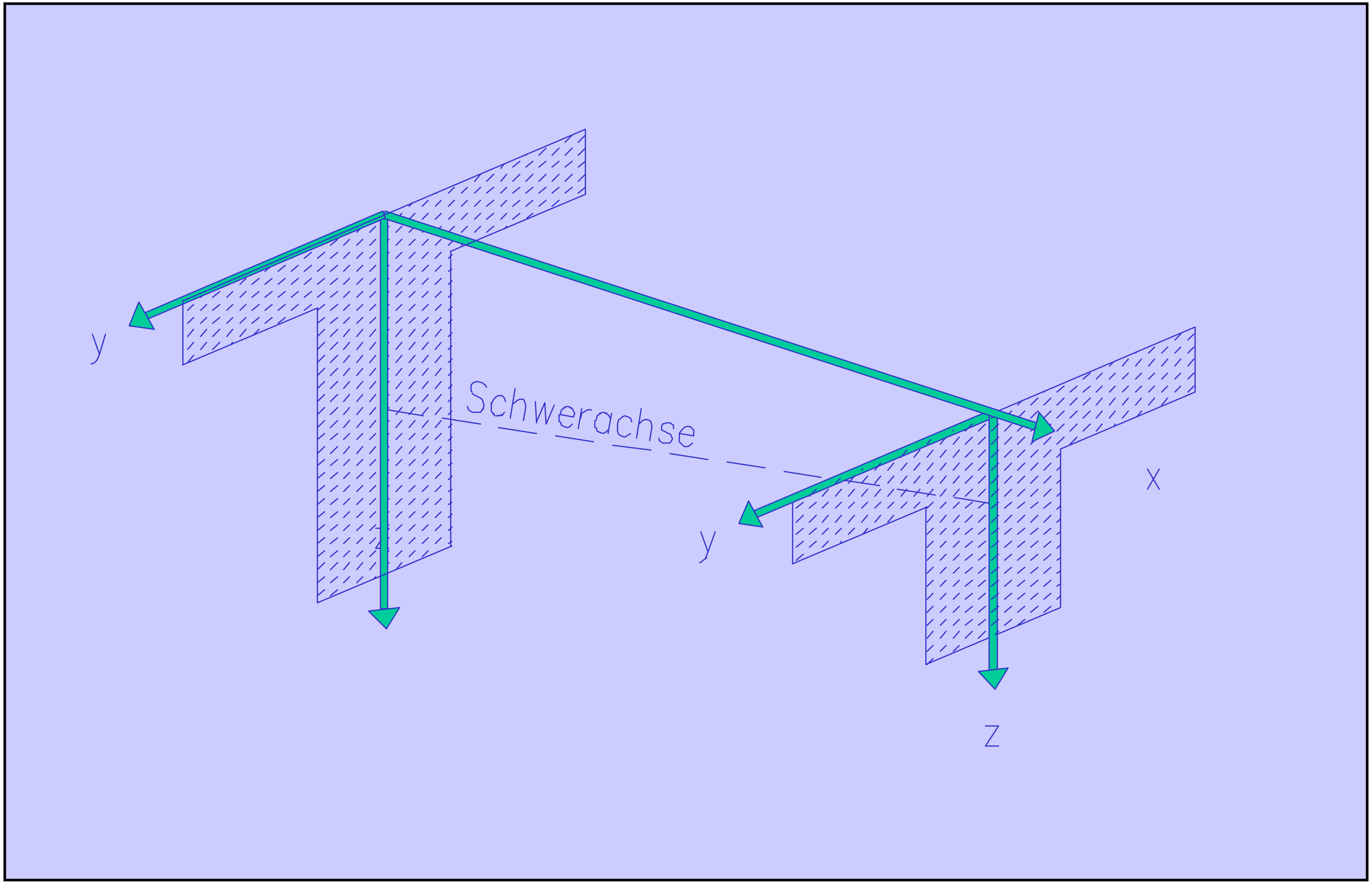



tau-nq in perpendicular section

# Shear in haunched beams

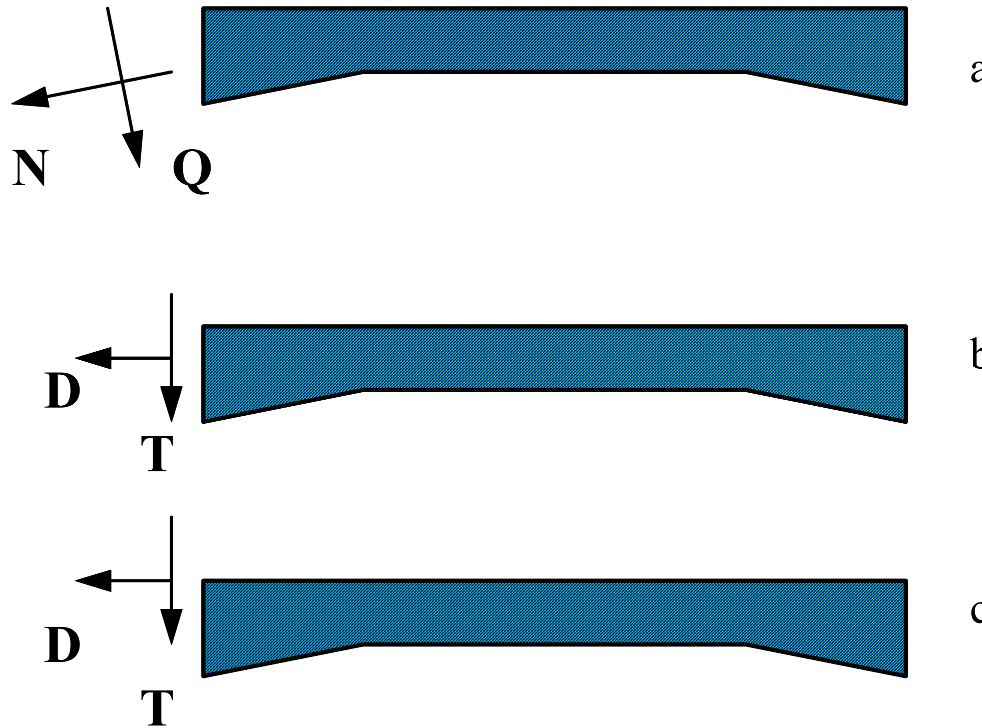
- Example for a shear reduction along the provisions of design codes
  - Shear force V — 40.0 kN
  - Yielding a faulty shear stress — 63.2 kN/m$^2$
  - Reduction of shear by M/d* tan$\alpha$ — 5.6 kN
  - Reduced shear stress — 54.9 kN/m$^2$
  - FE-shear stress — 53.0 kN/m$^2$
- Although the engineering approach has a completely different view, the maximum stress is quite good, distribution is faulty however.

# FE of a haunched beam



# Reference of forces ?

- Gravity axis with N + V
  - Results suited for design
- Gravity axis with D + T
  - Similar as for 2nd order Theory
  - Superposition of forces
- General reference axis
  - Superposition of forces if a composite section changes by in situ concrete or tendons.



# Principles of the FE-Beam

- Excentricities at the end points

$$u_{0i} = u_i + \varphi_{yi}\Delta z_i - \varphi_{zi}\Delta y_i$$
$$u_{0j} = uj + \varphi_{yj}\Delta z_j - \varphi_{zj}\Delta y_j$$

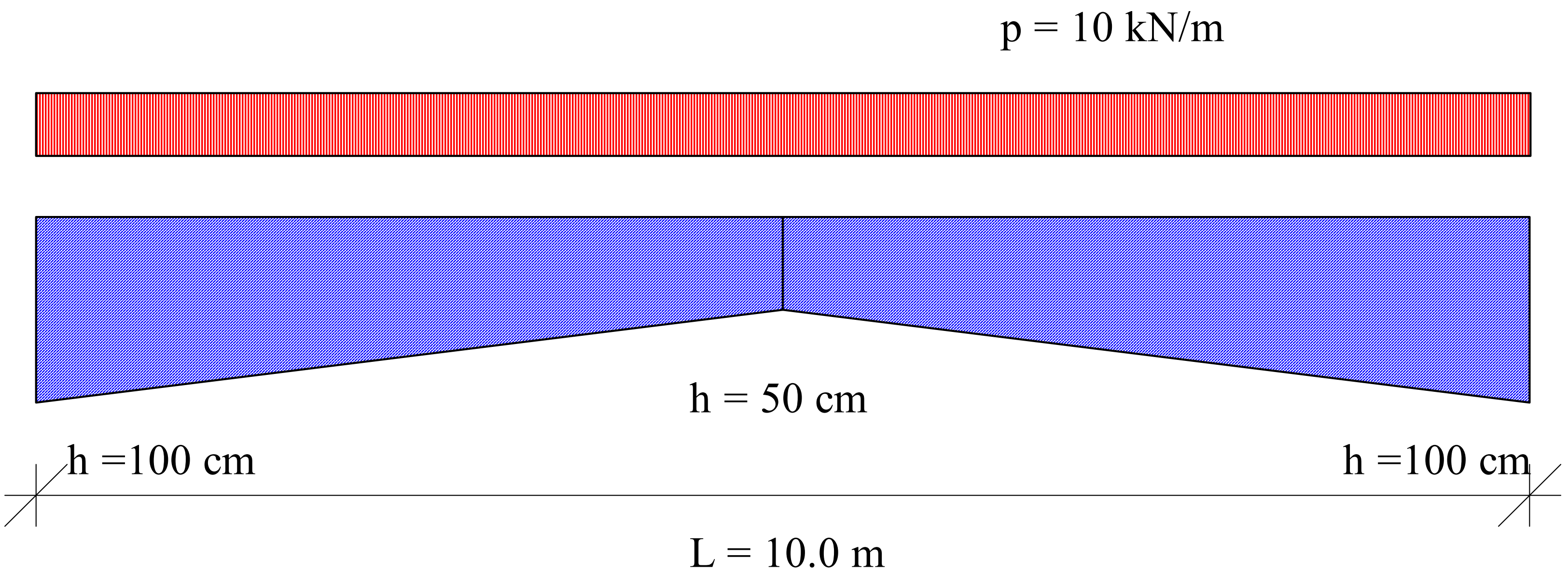
- Interpolation $u_0$ linear, v,w,$\varphi_x$,$\varphi_y$ cubic
- Displacements within section

$$u = u_0 + \varphi_y\left(z - z_s\right) - \varphi_z\left(y - y_s\right)$$

- Strains from derivative (position of gravity centre is not constant!)

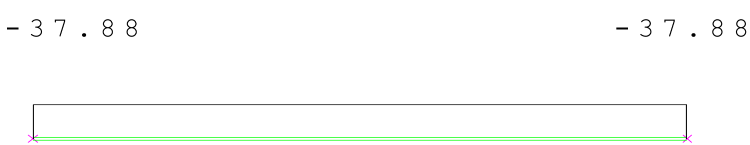
$$\varepsilon_x = \frac{\partial u}{\partial x} = u_o' + \varphi_y'\left(z - z_s\right) - \varphi_z'\left(y - y_s\right) - \varphi_y z_s' + \varphi_z y_s'$$

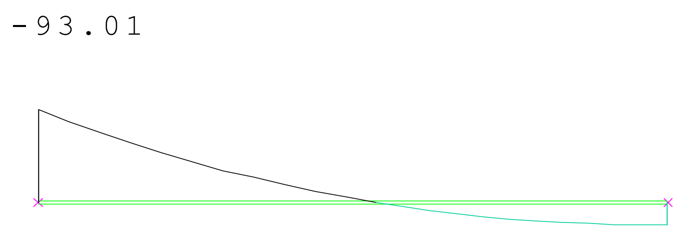
# Example of a haunched beam



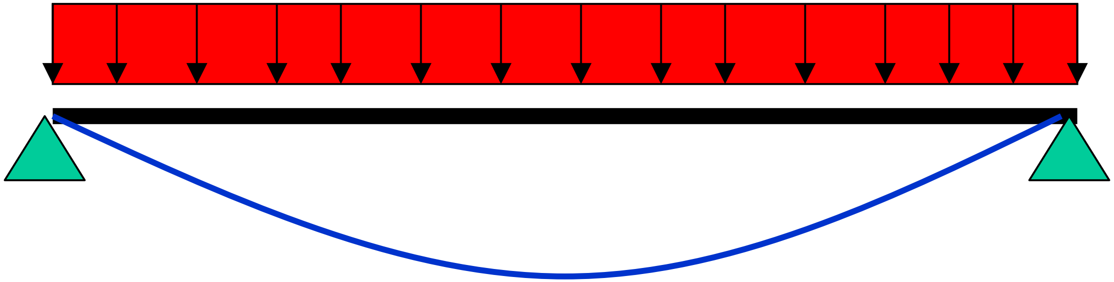
# Results for haunched beam

| | w[mm] | Ne[kN] | Nm[kN] | Mye[kNm] | Mym[kNm] |
|---|---|---|---|---|---|
| Inclined axis | | | | | |
| C-Beam (1 element) | 0,397 | -80,50 | -78,00 | -73.58 | 31,91 |
| C-Beam (8 elements) | 0.208 | -46,30 | -43,80 | -94,87 | 19,17 |
| FE-Beam (1 element) | 0,172 | -39,80 | -37,30 | -93,65 | 22,00 |
| Fe-Beam (8 elements) | 0.206 | -45,80 | -43,30 | -95,02 | 19,14 |
| Horiz. reference axis | | | | | |
| FE-Beam (1 element) | 0.168 | -37,90 | -37,90 | -93.01 | 22,52 |
| FE-Beam (8 elements) | 0.204 | -44,20 | -44,20 | -94.85 | 19,10 |



# The precision of the FE beam (bending)

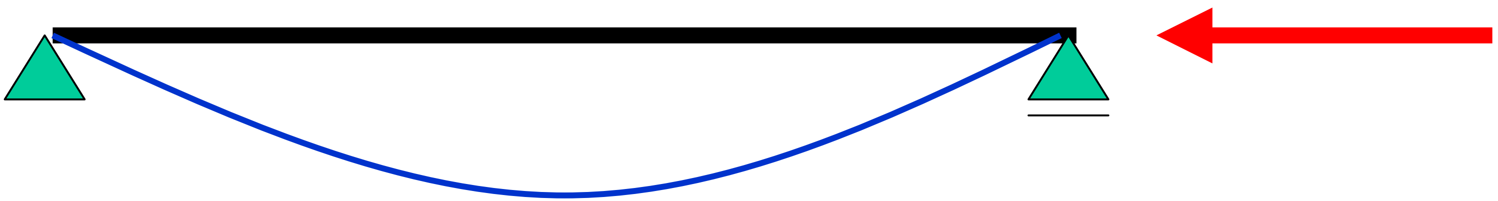


| | Exact | 1 Element | 2 Elements |
|---|---|---|---|
| Max. Moment | 281.25 | 281.25 | 281.25 |
| End Rotations | 41.147 | 41.147 | 41.147 |
| Cent. deflection | 19.2876 | 15.4301 | 19.2876 |

# Bending precision

- The theoretical solution is a polynom of 4th order
- The shape functions are cubical splines The highest symmetric ansatz function is the quadratic parabula
- As the influence function for the nodal displacements are cubic functions, the deformations and rotations in the nodes are exact.
- The moments and shear forces are also exact.
- The only values which are not are the displacements between the nodes (which are of less interest)
- For a buckling analysis there are the same principles, but know the buckling force (which has a high importance!) is not ok for a simple element.

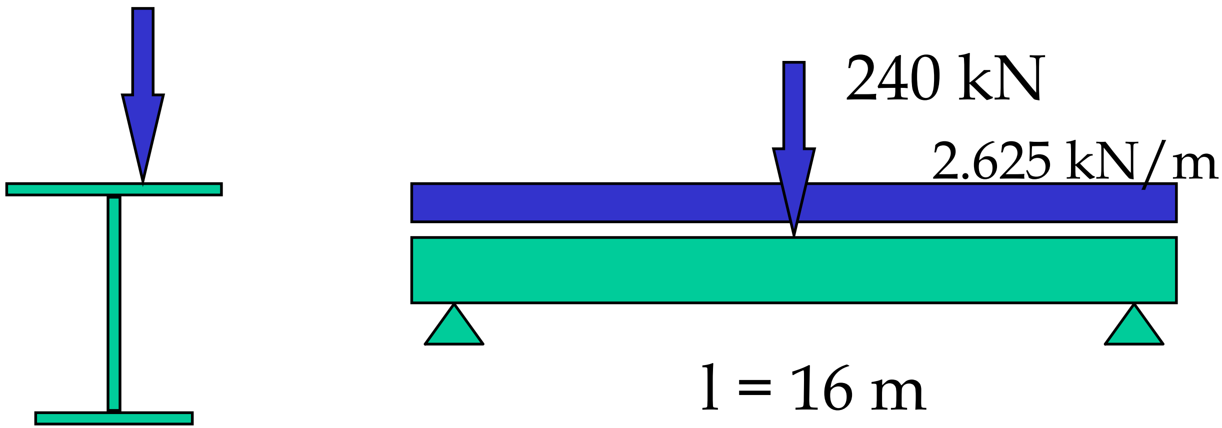
# The precision of the FE beam (Buckling analysis)



| | Euler II Exact | Euler II Numerical | Euler IV Exact | Euler IV Numerical |
|---|---|---|---|---|
| 1 Element | 13312 | 16065 | 52848 | - |
| 2 Elements | | 13312 | | 53550 |
| 3 Elements | | 13219 | | 53249 |
| 4 Elements | | 13213 | | 52879 |

# Warping Torsion and Shape deformation

- > 6. Degrees of freedom per node
- More forces and moments:
  secondary torsional moment Mt2 etc.




- Pure bending $\sigma$ = 84.3 N/mm²
- Warping / 2nd Order Torsion $\sigma$ = 136.1 N/mm²

# Conclusion

- Beam elements are not simple in theory
- Beam elements are easy to use
- There are a lot of modelling errors possible